MATH2050C Selected Solutions to Assignment 3

Deadline: Jan 31, 2018.

Hand in: Section 2.4 no 11, Section 2.5 no 14, Supp. Ex no 1.

Section 2.4 no. 8, 9, 10, 11.

(11) **Solution.** By definition, for each y,

$$\begin{aligned} g(y) &= \inf_x h(x,y) \\ &\le h(x,y)\ , \quad \forall x, \\ &\le \sup_z h(x,z), \quad \forall x, \\ &= f(x)\ , \quad \forall x. \end{aligned}$$

Taking infimum over all x, $g(y) \le \inf_x f(x)$. Now, taking supremum over y to get $\sup_y g(y) \le \inf_x f(x)$.

Section 2.5 no. 1, 2, 7, 8, 9, 14.

(14) **Solution.** WLOG assume $a_1 \neq b_1$ and $n \geq 2$. Then

$$\begin{aligned} \frac{1}{10} &\le \frac{|a_1 - b_1|}{10} \\ &\le \frac{|b_2 - a_2|}{10^2} + \cdots + \frac{|b_n - a_n|}{10^n} \\ &\le \frac{9}{10^2}\left(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-2}}\right) \\ &= \frac{1}{10}\left(1 - \frac{1}{10^{n-1}}\right) \end{aligned}$$

which is impossible.

Supplementary Exercises

1. Show that for every natural number $n \geq 2$ and every positive real number a, there is a positive real number b satisfying $b^n = a$. Suggestion: Modify the proof of $n = a = 2$ in the text book. Recall the binomial formula

$$(x+y)^n = x^n + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} y^k\ .$$

Solution. Let $S = \{x > 0 : x^n < a\}$. Claim S is bounded from above: Pick some $N > a$ by Archimedean property, then $x^n < a$ implies $x^n < N \leq N^n$, so $N^n - x^n > 0$. By factorization $(N^{n-1} + N^{n-2}x + \cdots + x^{n-1})(N - x) > 0$. Since the first factor is positive, $N - x > 0$, that is, N is an upper bound of S. By order-completeness, $b = \sup S$ exists. Next we show that $b^n < a$ is impossible. Assume that it is true and we draw a contradiction. Letting $1 > \varepsilon > 0$ be small, we have

$$(b+\varepsilon)^n = b^n + \sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^k = b^n + \varepsilon \sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^{k-1}\ .$$

Using

$$\sum_{k=1}^{n} \binom{n}{k} b^{n-k} \varepsilon^{k-1} \leq \sum_{k=1}^{n} \binom{n}{k} b^{n-k} \equiv c \ ,$$

$(b+\varepsilon)^n \leq b^n + c\varepsilon$. If we choose ε satisfies $\varepsilon < (a - b^n)/c$, then $(b+\varepsilon)^n < b^n + c\varepsilon < a$, contradicting the fact that b is the supremum of S. A similar argument shows that $b^n > a$ is also impossible, thus leaves the only case $b^n = a$.

3.1 Cardinality of Sets

For a finite non-empty set A, we define its power (cardinal number) to be the number of its elements. The power of an infinite set is not so easy to defined. First, we call two sets A and B have the same power if there is a bijective mapping. We call the power of A is less than or equal to B if there is an injective mapping from A into B. The power of A is less than the power of B if $|A| \leq |B|$ but there is no bijection between A and B. A nontrivial result is the following theorem.

Schröder-Bernstein Theorem If the power of A is less than or equal to that of B and vice versa, then there is a bijective map between A and B.

The equivalence class of all sets bijective to each other is called the power (or cardinal number) of the set. We will denote the power of a set by $|A|$. With these notations, Schroder-Bernstein theorem can be expressed as: $|A| \leq |B|$ and $|B| \leq |A|$ imply $|A| = |B|$.

Known facts:

- When A is finite, $|A|$ is the number of elements in A.
- For every infinite set A, $|\mathbb{N}| \leq |A|$. In other words, a countable set has the smallest infinity.
- $\mathbb{Z}, \mathbb{Q}$, etc, are all countable, that is, their powers are equal to $|\mathbb{N}|$.
- When A is finite, $|\mathcal{P}(A)| = 2^{|A|}$ where $\mathcal{P}(A)$ denotes the power set of A.
- For any set A, $|A| < |\mathcal{P}(A)|$.
- $|\mathbb{N}| < |\mathbb{R}|$.

We only give the proof concerning the power set. The last item is proved by the nested interval property in Text, and the others are elementary and left to you.

As $|A| < |\mathcal{P}(A)| = 2^{|A|}$ for any finite A, the power set always has more elements then the set itself when the set is finite. In general, assume on the contrary that there is bijection map Ψ from A to $\mathcal{P}(A)$. We let

$$A_1 = \{x \in A : \ x \in \Psi(x)\} \ , \quad A_2 = \{x \in A : \ x \text{ not in } \Psi(x)\} \ ,$$

so A is the disjoint union of A_1 and A_2. The set A_2 is nonempty, for, letting $a = \Psi^{-1}(\phi) \in A, \Psi(a) \in \phi$ means a does not belong to $\Psi(a)$, that is, $a \in A_2$. Now, let $z = \Psi^{-1}(A_2)$. Then either z belongs to A_2 or A_1. If $z \in A_2$, then z does not belong to $\Psi(z) = A_2$, impossible. On the other hand, if $z \in A_1$, then $z \in \Psi(z) = A_2$, again this is impossible. We conclude that there cannot have any bijective map between A and $\mathcal{P}(A)$. As the map $a \mapsto \{a\}$ is injective from A to $\mathcal{P}(A)$, we conclude $|A| < |\mathcal{P}(A)|$.

3.2 Decimal Representation of Real Numbers

We will show that every positive real number x can be represented by $m.m_2m_2m_3\cdots$ where $m \in \mathbb{N} \cup \{0\}, m_k \in D \equiv \{0, 1, 2, \cdots, 9\}$. More precisely,

Theorem For each positive real number x, there exist $m \in \mathbb{N}\cup\{0\}$, $m_k \in D \equiv \{0, 1, 2, \cdots, 9\}, k \geq 1$, such that x is the supremum of the sequence

$$\left\{m, m + \frac{m_1}{10}, \ m + \frac{m_1}{10} + \frac{m_2}{10^2}, \ m + \frac{m_1}{10} + \frac{m_2}{10^2} + \frac{m_3}{10^3}, \cdots, \right\} \ .$$

Proof: We can find a unique $m \geq 0$ such that $m \leq x < m+1$. Then

$$
\begin{aligned}
&0 \leq x - m < 1, \quad (0 \leq x - m < 1) \\
&0 \leq 10(x-m) < 10, \\
&\exists m_1 \in D, m_1 \leq 10(x-m) < m_1 + 1, \\
&0 \leq 10(x-m) - m_1 < 1, \quad (0 \leq x - m - \frac{m_1}{10} < \frac{1}{10}) \\
&0 \leq 10[10(x-m) - m_1] < 10, \\
&\exists m_2 \in D, m_2 \leq 10[10(x-m) - m_1] < m_2 + 1 \\
&0 \leq 10[10(x-m) - m_1] - m_2 < 1, \quad (0 \leq x - m - \frac{m_1}{10} - \frac{m_2}{10^2} < \frac{1}{10^2}) \ .
\end{aligned}
$$

Keep doing this, we get $m_k \in D, k \geq 1$, satisfying

$$0 \leq x - \left(m + \frac{m_1}{10} + \frac{m_2}{10^2} + \cdots + \frac{m_k}{10^k}\right) < \frac{1}{10^k},$$

and the conclusion follows easily. The proof is completed.

Introduce the notation

$$x = m.m_1 m_2 m_3 \cdots \ ,$$

and call the right hand side the decimal representation of x. Any number written in the form $m.m_1m_2m_3\cdots$ is called a decimal. You should always understand it stands for the supremum of an increasing sequence of rational numbers

$$\left\{m, m + \frac{m_1}{10}, \ m + \frac{m_1}{10} + \frac{m_2}{10^2}, \ m + \frac{m_1}{10} + \frac{m_2}{10^2} + \frac{m_3}{10^3}, \cdots, \right\} \ .$$

The decimal representation has the following properties:

- The decimal representations of two different numbers are different.
- A decimal $m.m_1m_2m_3\cdots$ satisfying $m_k = 9$ for all $k \geq k_0$ for some k_0 does not appear in the construction above. For instance, $0.999\cdots$ whose supremum is 1, but taking $x = 1$ in the proof gives $1.000\cdots$, not $0.999\cdots$. All other decimals come from some $x > 0$ though.
- Any decimal $m.m_1m_2m_3\cdots, m \geq 0, m_k \in D$, represents a rational number if and only if it is a repeated decimal.
- Replacing 10 by other natural number leads to other representations. It is called binary representation when 2 is used.

Prove them or google for more if you like.

3.3 The Real Line

We draw a line and put an arrow at the right end. Then fix a point called 0 and another called 1. Next mark $2, 3, 4, \cdots$ to right of 0 so that the lengths of the segment connecting n and $n+1$ are equal. Similarly, mark $-1, -2, -3, \cdots$, to the left of 0. In this way the real line is formed.

We claim that points on the real line are in one-to-one correspondence with the set of real numbers. We have done this already for all integers. For a rational number of the form p/q where p, q have no common factor greater than one and $0 < p < q$, we divide the segment connecting 0 and 1 into q many segments of equal length. Then the left endpoint of the p-th segment is p/q. When $p/q > 1$, write it as $n + p'/q$ where $p' < q$ and we move things to the segment connecting n and $n + 1$. Similarly we treat the negative case. Now, each irrational number is the supremum of a sequence of rational numbers. It is intuitively apparent that such sequence tends to a point on the real line and this point represents this irrational number. Conversely, it is clear that every point on the real line corresponds to a real number, thus accounting for the saying that the real line has no holes.